# Probability and Decoherence

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## How do Quantum Physics and Classical Physics relate?

- Decoherence is an attractive research program, but much work needs to be done to get a *detailed understanding* of the influence of the environment.
- My approach: Let's look at *specific aspects* of the relation between QP and CP and model these aspects.
- Note that in CP, there is always a joint probability distribution over a set of random variables.
- In QP, this is typically not the case (A. Fine, P. Suppes et al.).
- Hence, one aspect of the transition from QP to CP is the "emergence" of a joint probability distribution.
- But how does such a joint distribution emerge? And how long does it take?
- Plan: Study the emergence of a joint probability distribution as a result of decoherence.

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## Outline

1. Motivation

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2. GHZ states and joint probabilities
    - Bell states and GHZ states
    - Necessary and sufficient conditions

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   - Superoperators and superoperator algebras
   - Illustration of the method

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   - Bell states and GHZ states
   - Necessary and sufficient conditions
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   - Superoperators and superoperator algebras
   - Illustration of the method
4. The emergence of a joint distribution
   - The GHZ state and its evolution in time
   - Existence of a joint distribution

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   - Bell states and GHZ states
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4. The emergence of a joint distribution
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   - Existence of a joint distribution
5. Interpretation and open questions

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## Bell states and GHZ states

- Have a look at a (bipartite) Bell state such as the EPR state:

$$|\mathrm{EPR}>=\frac{1}{\sqrt{2}}\left(|10>-|01>\right)$$

- These states violate Bell's inequality, i.e. the correlations between the outcomes of measurements on these states cannot be accounted for by a non-contextual local hidden-variable model.
- Moreover, there is no joint probability distribution that returns the experimental probabilities.

# Bell states and GHZ states (cont'd)

GHZ states are tripartite states that lead too a straightforward *contradiction* if one assumes the existence of a non-contextual local hidden-variable model.

$$|\mathrm{GHZ}>=\frac{1}{\sqrt{2}}(|111>-|000>)$$

Define the following mutually commuting operators

$$\begin{array}{rcl} A & = & X_1Y_2Y_3 \quad , \quad B=Y_1X_2Y_3 \\ C & = & Y_1Y_2X_3 \quad , \quad D=X_1X_2X_3 \end{array}$$

with $X_1=\sigma_x^{(1)}$, $Y_1=\sigma_y^{(1)}$ etc.

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# A contradiction

In a GHZ state (which is an eigenstate of all four operators), these operators have the following expectation values:

$$E(A)=E(B)=E(C)=1 \text{ and } E(D)=E(-ABC)=-1.$$

This leads to a contradiction if we assume that there is one non-contextual hidden-variable $\lambda$ which is a function of the source and which determines the value of each of the $\pm 1$-spin components.

$$\begin{array}{lcl} m_{1x}(\lambda)m_{2y}(\lambda)m_{3y}(\lambda)=1 & , & m_{1y}(\lambda)m_{2x}(\lambda)m_{3y}(\lambda)=1 \\ m_{1y}(\lambda)m_{2y}(\lambda)m_{3x}(\lambda)=1 & , & m_{1x}(\lambda)m_{2x}(\lambda)m_{3x}(\lambda)=-1 \end{array}$$

Multiplying all four equations, one obtains: $1=-1$.

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# Necessary and sufficient conditions

de Barros and Suppes (2000) derived necessary and sufficient conditions for the existence of a joint probability distribution over the four random variables associated with the operators $A, B, C$ and $D$:

$$\begin{array}{rcl} -2 & \leq & E(A)+E(B)+E(C)-E(D)\leq 2 \\ -2 & \leq & -E(A)+E(B)+E(C)+E(D)\leq 2 \\ -2 & \leq & E(A)-E(B)+E(C)+E(D)\leq 2 \\ -2 & \leq & E(A)+E(B)-E(C)+E(D)\leq 2 \end{array}$$

Obviously, GHZ states violate the first equality. But GHZ states do not live forever. They experience the influence of decoherence and will eventually decay . . .

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# My main question

**Question**

Given that a GHZ state decays under the influence of decoherence, after what time will there be a joint probability distribution over the variables $A, B, C$ and $D$ that accounts for the measurement statistics?

Possible answers:

1. Only asymptotically
2. Already after a relatively short period of time (measured, for example, in units of the half-time of the decay)

To find out, we construct a model.

## Master equations and decoherence

- There is a whole range of models that allow us to study the influence of decoherence on a quantum state.
- Two major theoretical account are (i) Brownian motion (e.g. the Caldeira-Leggett model) and (ii) quantum optical master equations. I'll focus on the latter.
- We are interested in the influence of an environment (a reservoir or a heat bath) on a quantum state. Typically, this interaction will lead to a suppression (or "washing out") of quantum correlations.

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## Quantum optical master equations

A quantum system S is coupled to an environment B. This environment is modeled as an infinite collection of harmonic oscillators. Tracing the environment variables out (*pace* ignorance interpretation) and applying the Born-Markov approximation, one obtains an equation for the *reduced state* $P$ of the system S:

$$\frac{\partial}{\partial t}P = \frac{1}{i\hbar}[H, P] + LP \equiv \mathcal{L}P$$

Lindblad has shown that, under very general conditions, the non-unitary term $L$ has the form (with a bounded operator $O$)

$$LP = C\left(O^\dagger OP + PO^\dagger O - 2OPO^\dagger\right)$$

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## $Z$ atoms in a cavity

Let's assume that there are $Z$ atoms trapped in a quantum optical cavity with one field mode. The corresponding master equation is:

$$\frac{\partial}{\partial t}P = \frac{1}{i\hbar}[H^{(Z)}, P] + L_a P + L_\sigma^{(Z)} P \equiv LP$$

$H^{(Z)}$ is the Hamiltonian that governs the unitary part of the dynamics, i.e. the interaction of the atoms with the field mode (we work in the interaction picture).
$L_a$ and $L_\sigma^{(Z)}$ describe the influence of the environment on the field mode and on the $Z$ atoms respectively.

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## The unitary part: Jaynes-Cummings Hamilton operator

The unitary part is typically expressed in terms of the Jaynes-Cummings model:

$$H^{(Z)} = \hbar\sum_{i=1}^{Z}\left[g(a^\dagger\sigma_-^{(i)} + a\sigma_+^{(i)}) - \frac{1}{2}\Delta\ \sigma_3^{(i)}\right]$$

$a$ and $a^\dagger$ are the annihilation and creation operators of the field mode (with $[a, a^\dagger] = 1$) and the $\sigma$-matrices are the usual Pauli-matrices with their corresponding $SU(2)$-algebra acting on atom $i$:

$$[\sigma_+^{(i)}, \sigma_-^{(i)}] = 2\sigma_3^{(i)} \quad , \quad [\sigma_3^{(i)}, \sigma_\pm^{(i)}] = \pm\sigma_\pm^{(i)}$$

## The non-unitary part 1: field

$$L_a P = -\frac{A}{2}(\nu+1)\,[a^\dagger a P + P\,a^\dagger a - 2aPa^\dagger] - \frac{A}{2}\nu\,[aa^\dagger P + P\,aa^\dagger - 2a^\dagger P a]$$

- $A$ is a decay constant, and $\nu$ is the mean number of thermal photons in the cavity.
- Note that $L_a$ has the Lindblad form (although $a^\dagger$ is an unbounded oerator).

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## The non-unitary part 2: atoms

$$\begin{aligned} L_\sigma^{(Z)} P = & -\frac{B}{2}(1-s)\sum_{i=1}^{Z}[\sigma_+^{(i)}\sigma_-^{(i)}P + P\,\sigma_+^{(i)}\sigma_-^{(i)} - 2\sigma_-^{(i)}P\sigma_+^{(i)}] \\ & -\frac{B}{2}s\sum_{i=1}^{Z}[\sigma_-^{(i)}\sigma_+^{(i)}P + P\,\sigma_-^{(i)}\sigma_+^{(i)} - 2\sigma_+^{(i)}P\sigma_-^{(i)}] \\ & -\frac{2C-B}{4}\sum_{i=1}^{Z}[P - \sigma_3^{(i)}P\,\sigma_3^{(i)}] \end{aligned}$$

- $B, C$ are decay constants, $s$ is the pumping parameter.
- This operator models laser action ($s = 1$) and pure decays ($s = 0$), as well as processes in between these two extremes.
- This model works well, for example, for few-atom lasers.

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## Solving the master equation

- Solving the quantum optical master equation for $Z$ atoms is a difficult task. The field states are unbounded, and even the diagonalization of the pure atomic part is difficult as the number of basis states grows exponentially with $Z$.
- This is why various approximations are made, such as the introduction of quasi-probability distributions, which effectively wash out some of the correlations.
- These approximations, however, work well only for a large number of atoms (e.g. for laser systems), but not for systems with a small number of atoms or for mesoscopic systems with up to several hundred atoms. On the other hand, exact numerical simulations become computationally very expensive.
- Proposal: Develop an *analytical method*.

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## Superoperators . . .

We define the following superoperators:

$$\begin{aligned} \mathcal{Q}_{\alpha\beta}P &= \sum_{i=1}^{Z}\sigma_\alpha^{(i)}P\sigma_\beta^{(i)}\ ;\ i\in\{\pm,3\} \\ \mathcal{L}_\pm P &= \sum_{i=1}^{Z}\sigma_\pm^{(i)}P\ ;\quad \mathcal{L}_3 P = \frac{1}{2}\sum_{i=1}^{Z}\sigma_3^{(i)}P \\ \mathcal{R}_\pm P &= \sum_{i=1}^{Z}P\sigma_\pm^{(i)}\ ;\quad \mathcal{R}_3 P = -\frac{1}{2}\sum_{i=1}^{Z}P\sigma_3^{(i)} \end{aligned}$$

## and more superoperators . . .

$$\begin{array}{rcl} \mathcal{Q}_\pm & = & \mathcal{Q}_{\pm,\mp} \quad , \quad \mathcal{Q}_3 = \frac{1}{2}(\mathcal{L}_3 + \mathcal{R}_3) \\ \Sigma_\pm & = & \mathcal{Q}_{\pm,\pm} \quad , \quad \Sigma_3 = \frac{1}{2}(\mathcal{L}_3 - \mathcal{R}_3) \\ \mathcal{M}_\pm & = & \frac{1}{2}(\mathcal{Q}_{\pm 3} + \mathcal{L}_\pm) \quad , \quad \mathcal{M}_3 = \frac{1}{4}(\mathcal{Q}_{33} + 2\mathcal{L}_3) \\ \mathcal{N}_\pm & = & \frac{1}{2}(\mathcal{Q}_{\pm 3} - \mathcal{L}_\pm) \quad , \quad \mathcal{N}_3 = \frac{1}{4}(2\mathcal{L}_3 - \mathcal{Q}_{33}) \\ \mathcal{U}_\pm & = & \frac{1}{2}(\mathcal{Q}_{3\pm} + \mathcal{R}_\pm) \quad , \quad \mathcal{U}_3 = \frac{1}{4}(2\mathcal{R}_3 - \mathcal{Q}_{33}) \\ \mathcal{V}_\pm & = & \frac{1}{2}(\mathcal{Q}_{3\pm} - \mathcal{R}_\pm) \quad , \quad \mathcal{V}_3 = -\frac{1}{4}(\mathcal{Q}_{33} + 2\mathcal{R}_3) \end{array}$$

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## $SU(2)$ subgroups

- 15 of these 18 superoperators are independent. They are the generators of the Lie group $SU(4)$, and elements of the Lie algebra $su(4)$.
- $SU(4)$ has six $SU(2)$ subgroups:

$$\begin{array}{rcl} [\mathcal{Q}_+, \mathcal{Q}_-] & = & 2\mathcal{Q}_3 \\ [\mathcal{Q}_\pm, \mathcal{Q}_3] & = & \mp\mathcal{Q}_\pm \end{array}$$

- Analogous results obtain for $\Sigma$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{U}$ und $\mathcal{V}$.
- The $su(2)$-subalgebras $\mathcal{Q}$ and $\Sigma$ are "orthogonal":

$$[\mathcal{Q}_i, \Sigma_j] = 0 \quad \forall i, j \in \{\pm, 3\}$$

- Similarly for the subalgebras ($\mathcal{M}$ and $\mathcal{N}$) and ($\mathcal{U}$ and $\mathcal{V}$).

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## $SU(3)$ subgroups

$$\begin{array}{rcl} [\mathcal{Q}_+, \mathcal{M}_-] = \mathcal{V}_- & , & [\mathcal{Q}_+, \mathcal{M}_3] = \frac{1}{2}\mathcal{Q}_+ \\ [\mathcal{Q}_-, \mathcal{M}_+] = -\mathcal{V}_+ & , & [\mathcal{Q}_-, \mathcal{M}_3] = \frac{1}{2}\mathcal{Q}_- \\ [\mathcal{Q}_3, \mathcal{M}_+] = \frac{1}{2}\mathcal{M}_+ & , & [\mathcal{Q}_3, \mathcal{M}_-] = -\frac{1}{2}\mathcal{M}_- \\ [\mathcal{Q}_+, \mathcal{V}_+] = -\mathcal{M}_+ & , & [\mathcal{Q}_+, \mathcal{V}_3] = \frac{1}{2}\mathcal{Q}_+ \\ [\mathcal{Q}_-, \mathcal{V}_-] = \mathcal{M}_- & , & [\mathcal{Q}_-, \mathcal{V}_3] = -\frac{1}{2}\mathcal{Q}_- \\ [\mathcal{Q}_3, \mathcal{V}_+] = -\frac{1}{2}\mathcal{V}_+ & , & [\mathcal{Q}_3, \mathcal{V}_-] = \frac{1}{2}\mathcal{V}_- \end{array}$$

Simillarly for $(\mathcal{Q}, \mathcal{M}, \mathcal{V})$, $(\mathcal{Q}, \mathcal{N}, \mathcal{U})$, $(\Sigma, \mathcal{M}, \mathcal{U})$ and $(\Sigma, \mathcal{N}, \mathcal{V})$.

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## Basis states and quantum numbers

- To construct the basis states, a symmetry type $Y$ has to be specified first. More on this below.
- The basis states of our "noise algebra" can then be characterized by four quantum numbers:
  1. $Z$: number of atoms
  2. $q : \mathcal{Q}^2 \; P^{(Z)}_{q,q_3,\sigma_3} = q(q+1) \; P^{(Z)}_{q,q_3,\sigma_3} \quad (q = 0, \frac{1}{2}, \ldots, K_s/2)$
  3. $q_3 : \mathcal{Q}_3 \; P^{(Z)}_{q,q_3,\sigma_3} = q_3 \; P^{(Z)}_{q,q_3,\sigma_3} \quad (-q \leq q_3 \leq q)$
  4. $\sigma_3 : \Sigma_3 \; P^{(Z)}_{q,q_3,\sigma_3} = \sigma_3 \; P^{(Z)}_{q,q_3,\sigma_3} \quad (-\sigma \leq \sigma_3 \leq \sigma)$
- One can show that

$$q + \sigma = K_s/2 \quad (K_s = K_s(Y))$$

# The fundamental representation of $SU(4)$

$$u := |1 >< 1| \quad , \quad d := |0 >< 0|$$
$$s := |1 >< 0| \quad , \quad c := |0 >< 1|$$

| | $Z$ | $q$ | $q_3$ | $\sigma$ | $\sigma_3$ |
|---|---|---|---|---|---|
| u | 1 | 1/2 | 1/2 | 0 | 0 |
| d | 1 | 1/2 | - 1/2 | 0 | 0 |
| s | 1 | 0 | 0 | 1/2 | 1/2 |
| c | 1 | 0 | 0 | 1/2 | -1/2 |

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# Higher dimensional representations

Higher dimensional representations follow from the fundamental representation once a symmetry type $Y$ is specified.

⇒ Young tableaux

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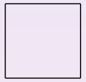
# The fundamental representation



Example $SU(2)$: There are two basis states

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow> \quad , \; v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow>$$

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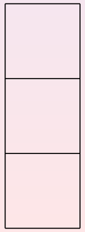
# Higher dimensional representations

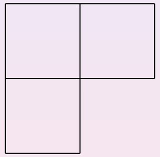
a) Fully symmetrical states

b) Fully antisymmetrical states



# Young tableaux

c) States of mixed symmetry



- A tableaux of the group $SU(N)$ has maximally $N$ rows.
- Classification of Young tableaux: $p_1, p_2, \ldots p_N$; $p_i = \sharp$ boxes in row $i$
- Alternatively: $\lambda_1, \ldots \lambda_{N-1}; \lambda_i = p_i - p_{i+1}$

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# Dimension of a representation and construction of states

There is a very simple algorithm to calculate the dimension of a tableau ("hook rule").

Construction of states:

Example 1:

$$P = \mathcal{S}_{12}\ (ud) = ud + du$$

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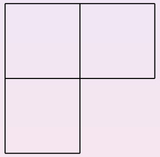
# Example 2



$$P = \mathcal{A}_{13}\ \mathcal{S}_{12}\ (uds) = \mathcal{A}_{13}\ (uds + dus) = uds - sdu + dus - sud$$

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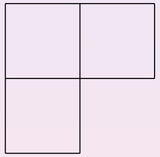
# Example 3



$$P = \mathcal{A}_{12}\ \mathcal{S}_{13}\ (uds) = \mathcal{A}_{12}\ (uds + sdu) = uds - dus + sdu - dsu$$

## Arbitrary representations

- General formula:

$$P^{(Z)}_{q,q_3,\sigma_3} = \mathcal{S}^Y\ (u^\alpha d^\beta s^\gamma c^\delta)$$

  $\alpha, \beta, \gamma$ and $\delta$ can be obtained from $Z, q, q_3, \sigma_3$ and $Y$.
- Important relation:

$$q + \sigma = \frac{K_s}{2}$$

$$K_s = p_1 + p_2 - p_3 - p_4 = \lambda_1 + 2\lambda_2 + \lambda_3$$

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## Other possibility to construct the basis states

Start with a state of *highest weight* given a certain symmetry type $Y$ (e.g. a fully symmetrical state with $P = u^Z, d^Z, s^Z$ or $c^Z$ for $p = (Z, 0, 0, 0)$) and apply the raising and lowering operators $\mathcal{Q}_\pm$ etc. successively.

Helpful relations:

| | $u$ | $d$ | $s$ | $c$ |
|---|---|---|---|---|
| $\mathcal{Q}^i_+$ | 0 | u | 0 | 0 |
| $\mathcal{Q}^i_-$ | d | 0 | 0 | 0 |
| $\mathcal{Q}^i_3$ | $u/2$ | $d/2$ | 0 | 0 |

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## Are non-symmetrical states physically interesting?

- Realisiable if the distance between the atoms is larger than the de Brogie-wavelength of the atoms.
- Conjecture: non-symmetrical admixtures (trace $= 0$) are damped out quickly over time.
- Until they are damped out, admixed non-symmetrical states change the atomic coherences.

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## Completely symmetrical states

- Everything becomes very simple!
- The basis states are given by

$$P^{(Z)}_{q,q_3,\sigma_3} = \frac{1}{N}\ \mathcal{S}\ (u^\alpha d^\beta s^\gamma c^\delta)$$

  with $\alpha = q + q_3, \beta = q - q_3, \gamma = \sigma + \sigma_3, \delta = \sigma - \sigma_3$ and

$$N = \begin{pmatrix} Z \\ \alpha, \beta, \gamma, \delta \end{pmatrix}$$

- Example:

$$\begin{aligned} P^{(3)}_{1,1,1/2} &= \frac{1}{3}\ \mathcal{S}\ (u^2 s) = \frac{1}{3}\ (u^2 s + usu + su^2) \\ &= \frac{1}{3}\ |111 > (< 110| + < 101| + < 011|) \end{aligned}$$

## Interpretation

- $2q$: Number of coincidences on the LHS and the RHS of $P$
  $2\sigma$: Number of non-coincidences on the LHS and RHS of $P$
- The eigenvalues are given by

$$\begin{aligned} n_+^L &= 2v = Z/2 + q_3 + \sigma_3 \\ n_-^L &= 2u = Z/2 - q_3 - \sigma_3 \\ n_+^R &= 2m = Z/2 + q_3 - \sigma_3 \\ n_-^R &= 2n = Z/2 - q_3 + \sigma_3 \end{aligned}$$

- Here $n_+^L$ denotes the number of 1s on the LHS of $P$, $n_-^R$ denotes the number of 0s on the RHS of $P$, etc.
- Note that $m + n + u + v = Z$ and $q + \sigma + m + n + u + v = 3/2Z$ (for fully symmetrical states).

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## An alternative way to construct the basis . . .

. . . for fully symmetrical states:

$$P^{(Z)}_{q,q_3,\sigma_3} = \frac{1}{N}\, \mathcal{S}_{[2q]} \left( |\{n_+^L, n_-^L\} >< \{n_+^R, n_-^R\}| \right)$$

with

$\mathcal{S}_{[2q]}$ (= conditionalized symmetrizer): Symmetrize the left and the right part of the density operator with the constraint that the number of coincidences is $2q$.

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## Example: calculate $P^{(3)}_{1/2,1/2,0}$

In this case, we have $Z = 3, q = \frac{1}{2}, q_3 = \frac{1}{2}, \sigma_3 = 0$.

Hence, $n_+^L = 2, n_-^L = 1, n_+^R = 2, n_-^R = 1$.

We then obtain:

$$\begin{aligned} P^{(3)}_{1/2,1/2,0} &= \frac{1}{6}\, \mathcal{S}_{[1]} \left( |\{110\} >< \{101\}| \right) \\ &= \frac{1}{6} \left( |110 >< 101| + |110 >< 011| \right. \\ &+ |101 >< 110| + |101 >< 011| \\ &+ \left. |011 >< 101| + |011 >< 110| \right) \end{aligned}$$

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## Some properties

- Dimension of the basis for given number of atoms $Z$ (and 2 atomic levels):

$$D(Z,2) = \frac{1}{6}(Z+1)(Z+2)(Z+3)$$

  $(D(5,2) = 56, D(10,2) = 286, D(20,2) = 1771)$

  This has to be compared with $4^Z$ if one proceeds by brute force, e.g. $4^{10} \approx 3670 \times 286$.
- Dual states

$$\check{P}^{(Z)}_{q,q_3,\sigma_3} = P^{(Z)}_{q,q_3,-\sigma_3}$$

- Biorthogonality

$$tr(\check{P}^{(Z)}_{q,q_3,\sigma_3} P^{(Z)}_{q',q'_3,\sigma'_3}) = \delta_{ZZ'}\ \delta_{qq'}\ \delta_{q_3q'_3}\ \delta_{\sigma_3\sigma'_3}$$

## Eigenvalues of the Lindblad operator

Idea: Solve the eigenvalue equation

$$[L, X_\alpha] = \rho(\alpha)\, X_\alpha$$

with

- $X_\alpha$: Raising and lowering operators for the eigenstates of $L$
- $\rho(\alpha)$: Difference of eigenvalues of $L$

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## Eigenvalues of the Lindblad operator (cont'd)

Express the atomic part of the Liouville operator and $X_\alpha$ in terms of the fifteen $SU(4)$ generators $Y_i$ (i.e. $\mathcal{Q}_\pm$ etc.):

$$L_\sigma = \sum_{i=1}^{15} y_i Y_i \quad , \quad X_\alpha = \sum_{i=1}^{15} x_i^\alpha Y_i$$

Use the $SU(4)$ commutator relations:

$$[Y_i, Y_j] = f_{ijk}\, Y_k$$

Pluging this in $[L, X_\alpha] = \rho(\alpha)\, X_\alpha$ we are left with

$$\det\ [y_i f_{ijk} - \rho(\alpha)\delta_{jk}] = 0,$$

which we can solve numerically to obtain the eigenvalues and eigenvectors of $L$.

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## Example: eigenvalues of the damping

$$\begin{aligned} L_\sigma P = & -\frac{B}{2}(1-s)\sum_{i=1}^{Z}[\sigma_+^{(i)}\sigma_-^{(i)}P + P\,\sigma_+^{(i)}\sigma_-^{(i)} - 2\sigma_-^{(i)}P\sigma_+^{(i)}] \\ & -\frac{B}{2}s\sum_{i=1}^{Z}[\sigma_-^{(i)}\sigma_+^{(i)}P + P\,\sigma_-^{(i)}\sigma_+^{(i)} - 2\sigma_+^{(i)}P\sigma_-^{(i)}] \end{aligned}$$

Express the right hand side in terms of the $SU(4)$ generators:

$$L_\sigma P = \left[-\frac{1}{2}BZ - B(1-2s)\ \mathcal{Q}_3 + B(1-s)\ \mathcal{Q}_- + Bs\ \mathcal{Q}_+\right] P$$

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## Eigenvalues of the damping (cont'd)

In this case, only a $3 \times 3$-matrix has to be diagonalised. One obtains $\rho = 0, \pm 1$. From this, all eigenvalues of the damping can be obtained:

$$\lambda_i^{(Z,Q)} = \left(-\frac{Z}{2} - Q + i\right) B \quad , \quad 0 \le i \le 2Q$$

So if one knows, for example, the stationary state $P_{ss}$ of the system, then all other states can be obtained by applying the ladder operators $X_\pm$.

## Exact solutions

$$\frac{\partial}{\partial t} P = LP \Rightarrow P(t) = \exp(Lt)P(0)$$

Idea: As

$$L = \sum_{i=1}^{15} y_i Y_i$$

$\exp(Lt)$ can be obtained by using BCH-type formulas:

$$P(t) = \prod_{i=1}^{15} \exp(x_i Y_i) P(0)$$

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## The emergence of a joint distribution

Let's consider the case of pure decoherence. In this case, the atoms do not couple to the field, the pumping parameter $s = 0$, and the Liouville operator has the following form:

$$L = (-Z/2 + \mathcal{Q}_- - \mathcal{Q}_3)\, B$$

The solution of the master equation

$$\frac{\partial}{\partial t} P = LP$$

is hence

$$P(t) = e^{-Z/2\tau} e^{(\mathcal{Q}_- - \mathcal{Q}_3)\tau}\; P(0)$$

with $\tau = Bt$. Using BCH-like tricks, we obtain

$$P(t) = e^{-Z/2\tau} e^{-\tau \mathcal{Q}_3}\; e^{f(\tau)\mathcal{Q}_-}\; P(0)$$

with $f(\tau) = 1 - e^{-\tau}$.

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## The GHZ state in our basis

The GHZ state can be expressed in our fully symmetrical basis for $Z = 3$. We obtain:

$$\begin{aligned} P_{GHZ} &= |GHZ >< GHZ| = 1/2\left(|111 >< 111| + |000 >< 111|\right. \\ &- \left.|111 >< 000| - |000 >< 111|\right) \\ &= 1/2\left(P_{3/2,3/2,0} + P_{3/2,-3/2,0} - P_{0,0,3/2} - P_{0,0,-3/2}\right) \end{aligned}$$

We can now use this state as our $P(0)$ and propagate it through time.

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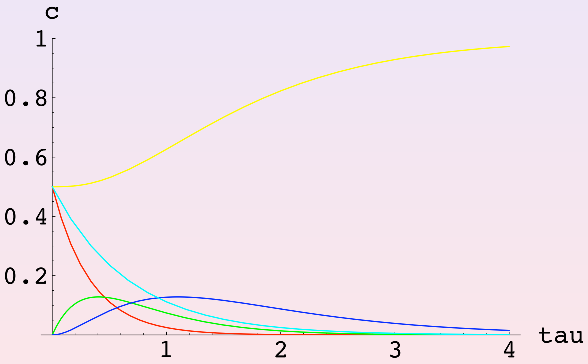
## The time evolution of the GHZ state

$$\begin{aligned} P_{GHZ}(\tau) = {} & c_1(\tau)P_{3/2,3/2,0} + c_2(\tau)P_{3/2,1/2,0} + c_3(\tau)P_{3/2,-1/2,0} \\ & + c_4(\tau)P_{3/2,-3/2,0} - c_5(\tau)\left(P_{0,0,3/2} + P_{0,0,-3/2}\right) \end{aligned}$$

with

$$\begin{aligned} c_1(\tau) &= 1/2 e^{-3\tau} \\ c_2(\tau) &= 3/2\left(e^{-2\tau} - e^{-3\tau}\right) \\ c_3(\tau) &= 3/2\left(e^{-\tau} - 2e^{-2\tau} + e^{-3\tau}\right) \\ c_4(\tau) &= 1 - 1/2\left(3e^{-\tau} - 3e^{-2\tau} + e^{-3\tau}\right) \\ c_5(\tau) &= 1/2 e^{-3/2\tau} \end{aligned}$$

## Plot of $c_1(\tau), \ldots, c_5(\tau)$



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## Existence of a joint distribution

- Recall the operators:

$$\begin{aligned} A &= X_1 Y_2 Y_3 \quad , \quad B = Y_1 X_2 Y_3 \\ C &= Y_1 Y_2 X_3 \quad , \quad D = X_1 X_2 X_3 \end{aligned}$$

- We calculate the time evolution of the expectation values of these operators in the (evolving) GHZ state.

$$< A(\tau) >= Tr(P(\tau)A) \ etc.$$

- Note that the only non-vanishing contributions come from the states $P_{0,0,\pm 3/2}$ as the operators $A, B, C$ and $D$ change a 1-state into a 0-state and vice versa.
- Insert in necessary and sufficient conditions: A joint probability distribution over $A, B, C$ and $D$ exists if $\tau > \tau_0 = .46$, i.e. if $t > .46 \times B^{-1}$.

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## Further remarks

1. The joint distribution can be constructed by solving a coupled system of 16 equations that use equations of the form

$$E(A) = p_{AB} - p_{A\overline{B}} - p_{\overline{A}B} + p_{\overline{AB}}$$

with the expectation values calculated quantum mechanically.

2. It turns out that for $\tau < \tau_0$, there exists an upper probability distribution.

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## Interpretation and open questions

- Note that $\tau_0$ is rather small. The state is still highly entangled, and yet, a classical model can account for the statistics.
- But: How stable are these results? Do they hold for other observables as well? And for other states?
- Can we construct states that have a much greater decoherence time? This would be interesting for applications in quantum computing. Look at noiseless subsystems.
- What happens if the interaction with a radiation field is taken into account?